ON LINEAR DIFFERENTIAL EQUATIONS WITH EXPONENTIAL COEFFICIENTS AND STATIONARY DELAYS OF THE ARGUMENT. REGULAR CASE

(O LINEINYKH DIFFERENTSIAL'NYKH URAVNENIIAKH S EKSPONENTSIAL'NYMI KOEFFITSIENTAMI I STATSIONARNYMI ZAPAZDYVANIIAMI ARGUMENTA REGULIARNYI SLUCHAI)

PMM Vol.26, No.3, 1962, pp. 449-454

K.G. VALEEV
(Leningrad)

(Received February 5, 1962)

The equations with exponential coefficients and with stationary delays [lags] in the argument that are considered here, are frequently met in engineering problems. The present investigation is carried out by a method which is a generalization of the one used by the author previously [1]. The problem is reduced to the study of the Laplace transform [2] of the solution of a system of differential equations. This solution is obtained in an asymptotic form for large values of the argument.

The method presented makes it possible to construct a particular solution satisfying certain initial conditions. The construction of the solution of the system of linear equations in the neighborhood of a regular singular point differs from Frobenius' method in a way similar to the one in which Euler's method, for the solution of linear differential equations with constant coefficients, differs from the solution of these equations by the use of the Laplace transform [2].

1. The following system of linear differential equations, with exponential coefficients and stationary delays in the argument, is considered

$$\sum_{q=0}^{l} e^{-\alpha_q t}\left(A_{qn}\frac{d^nY(t)}{dt^n}+\sum_{k=0}^{n-1}\int_{-h}^{0} dA_{qk}(\vartheta)\frac{d^kY(t+\vartheta)}{dt^k}\right)=\Phi(t) \qquad (1.1)$$

Here, $Y(t)$ is an m-dimensional vector, A_{qn} are complex $m \times m$ matrices satisfying the conditions

$$A_{0n} \equiv E, \qquad \sum_{q=1}^{l} |A_{qn}| \leqslant \mu_0 < 1 \tag{1.2}$$

where E is the identity matrix. The symbol $|A|$ denotes the norm of the matrix

$$A = \|a_{sj}\|_1^m, \qquad |A| = \max_s \sum_{j=1}^{m} |a_{sj}| \tag{1.3}$$

The elements $a_{sj}^{qk}(\vartheta)$ of the matrix $A_{qk}(\vartheta) = \| a_{sj}^{qk}(\vartheta) \|_1^m$ are functions of bounded variation on $[-h, 0]$, $(h > 0)$ [3]. The number l in (1.1) is usually assumed to be finite. The case when $l = \infty$ will be considered separately. Here we shall assume that

$$\sum_{q=0}^{\infty} |\alpha_q|^n \, |A_{qn}| \leqslant c, \qquad \sum_{q=0}^{\infty} \overset{0}{\underset{-h}{V}} \, a_{sj}^{qk}(\vartheta) \, |\alpha_q|^k \leqslant c = \text{const} \tag{1.4}$$

$$(k = 0, 1, \ldots, n-1; \; s, j = 1, \ldots, m)$$

The differential in front of the matrices $A_{qk}(\vartheta)$ in (1.1) are Stieltjes integrals [3, p.277].

The complex numbers α_q satisfy the following conditions

$$\alpha_0 \equiv 0, \qquad \operatorname{Re} \alpha_q \geqslant 0 \qquad (q = 1, 2, \ldots, l) \tag{1.5}$$

Among the numbers $\operatorname{Im} \alpha_q$ there can be rationally noncommensurate numbers. Suppose that the transform of the vector $\Phi(t)$ $(t > 0)$ is a meromorphic vector $Q(p)$ whose components are regular and bounded when $\operatorname{Re} p > b = \text{const}$.

In a particular case we shall assume that

$$\Phi(t) = \sum_{j=1}^{\lambda} C_j t^{\nu_j} e^{\omega_j t}, \qquad Q(p) = \sum_{j=1}^{\lambda} C_j \nu_j! \, (p - \omega_j)^{-\nu_j - 1} \tag{1.6}$$

where C_j are constant complex vectors, ν_j are non-negative integers, and the ω_j are complex numbers. With $t > 0$, we shall seek the system's (1.1) solution $Y(t)$ satisfying the initial conditions

$$Y(t) = Y_0^{(0)}(t), \ldots, \frac{d^{n-1} Y(t)}{dt^{n-1}} = Y_0^{(n-1)}(t), \qquad t \in [-h, 0] \tag{1.7}$$

where the vectors $Y_0^{(j)}$ $(j = 0, 1, \ldots, n-1)$ satisfy Dirichlet's condition when $t \in [-h, 0]$. We assume that the vectors $Y(t), \ldots, d^{n-1}Y(t)/dt^{n-1}$ are continuous from the right at the point $t = 0$.

2. In this section there is constructed a system of linear difference

equations for the image $F(p)$ of the solution $Y(t)$. The solution of this system of linear difference equations is obtained in the form of a series of matrices. The relationship between the image $F(p)$ and the original function $Y(t)$ will be denoted by a small arrow as follows

$$Y(t) \leftarrow F(p), \qquad F(p) = \int_0^\infty Y(t)\, e^{-pt}\, dt \tag{2.1}$$

Multiplying the system (1.1) by e^{-pt} and integrating with respect to t from 0 to ∞, we obtain for $F(p)$ the system of linear difference equation [1]

$$\sum_{q=0}^{l} L_q(p+\alpha_q) F(p+\alpha_q) = R(p) \tag{2.2}$$

Here

$$L_q(p) = A_{qn}p^n + \sum_{k=0}^{n-1} p^k \int_{-h}^{0} e^{p\vartheta}\, dA_{qk}(\vartheta) \qquad (q = 0, 1, \ldots, l) \tag{2.3}$$

$$R(p) = Q(p) + \sum_{q=0}^{l} \Psi_q(p+\alpha_q) \qquad (Q \text{ is given in } (1.6)) \tag{2.4}$$

$$\Psi_q(p) = A_{qn} \sum_{j=0}^{n-1} Y_0^{(j)}(0)\, p^{n-j-1} + \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} \int_{-h}^{0} e^{p\vartheta}\, dA_{qk}(\vartheta)\, Y_0^{(j)}(0)\, p^{k-j-1} - $$
$$- \sum_{k=0}^{n-1} \int_{-h}^{0} \int_{t}^{0} e^{p(\vartheta-t)}\, dA_{qk}(\vartheta)\, Y_0^{(k)}(t)\, dt \tag{2.5}$$

The elements of the known matrices $L_q(p)$ are entire functions of p which satisfy the following condition uniformly in $\operatorname{Im} p$

$$\overline{\lim_{p\to\infty}} \sum_{q=1}^{n} |L_0^{-1}(p) L_q(p+\alpha_q)| \leqslant \mu < 1, \qquad \operatorname{Re} p \geqslant b = \text{const} \tag{2.6}$$

This is implied by (1.2).

Here $R(p)$ is a known vector, and we have, in view of (2.4),

$$\lim |L_0^{-1}(p) R(p)| = 0, \qquad \operatorname{Re} p \to +\infty \tag{2.7}$$

Making use of (2.3) and (2.4) we introduce the notation

$$K_q(p) = -L_0^{-1}(p) L_q(p+\alpha_q), \qquad \Omega(p) = L_0^{-1}(p) R(p) \tag{2.8}$$

If the system of difference equations (2.2) is multiplied by $L_0^{-1}(p)$

and solved for $F(p)$ one obtains

$$F(p)=\sum_{q=1}^{n} K_q(p)F(p+\alpha_q)+\Omega(p) \tag{2.9}$$

When $\operatorname{Re} p \geqslant b_1$, where b_1 is sufficiently large, the relation (2.9) can be considered as a contraction mapping [3, p.44] of the space of bounded regular (when $\operatorname{Re} p \geqslant b_1$) vector functions. The metric in this space is defined as

$$\rho(F_1(p),\ F_2(p))=\max_j\{|f_{1j}(p)-f_{2j}(p)|\},\qquad \operatorname{Re} p\geqslant b_1 \tag{2.10}$$

where the $f_{sj}(p)$ are the components of the vector $F_s(p)$ ($s = 1, 2$). The system's (2.2) solution $F(p)$, which is bounded for sufficiently large $\operatorname{Re} p \geqslant b_1$, is unique and can be obtained by the method of successive approximations [3, p.45]. We have

$$F_0(p)\equiv 0,\qquad F_{j+1}(p)=\sum_{q=1}^{l} K_q(p)F_j(p+\alpha_q)+\Omega(p)\qquad (j=0,1,2,\ldots) \tag{2.11}$$

The sequence of functions $F_j(p)$ converges uniformly to the vector $F(p)$, which is regular when $\operatorname{Re} p \geqslant b_1$, because

$$\sum_{q=1}|K_q(p)|\leqslant\mu_1<1\quad \text{when } \operatorname{Re} p\geqslant b_1 \tag{2.12}$$

From (2.11) we obtain the following expression for $F(p)$

$$\begin{aligned}F(p)=\Omega(p)+\sum_{\sigma=1}^{\infty}\ \sum_{q_j=1,2,\ldots,l} &K_{q_1}(p)K_{q_2}(p+\alpha_{q_1})K_{q_3}(p+\alpha_{q_1}+\alpha_{q_2})\ldots\\ &\ldots K_{q_\sigma}(p+\alpha_{q_1}+\alpha_{q_2}+\ldots+\alpha_{q_{\sigma-1}})\,\Omega(p+\alpha_{q_1}+\alpha_{q_2}+\ldots+\alpha_{q_\sigma})\end{aligned} \tag{2.13}$$

3. Regular case $\alpha_0 = 0$, $\operatorname{Re}\alpha_q > 0$ ($q = 1, \ldots, l$). In this case the system (1.1) is especially simple. Let us consider the equation which we shall call the generating one of (2.3)

$$\operatorname{Det} L_0(p)=0 \tag{3.1}$$

We denote its roots by $\rho_0, \rho_1, \rho_2, \ldots, \rho_k, \ldots$ and introduce into our consideration the numbers $p_{k_0,\ k_1,\ \ldots,\ k_l}$, where

$$p_{k_0,k_1,\ldots,k_l}=\rho_{k_0}-k_1\alpha_1-k_2\alpha_2-\ldots-k_l\alpha_l\quad (k_q=0,1,2,\ldots;\ q=0,1,\ldots,l) \tag{3.2}$$

Next we designate by Σ_ε the multiply-connected region of the complex plane p defined by the inequality

$$|p - p_{k_0, k_1, \ldots, k_l}| \geqslant \varepsilon > 0 \qquad (k_q = 0, 1, 2, \ldots, \; q = 0, 1, \ldots, l) \tag{3.3}$$

If $p \in \Sigma_\varepsilon$ and Re $p \geqslant b_2 =$ const, then it follows from (2.12) and (2.6) that the series (2.13) converges absolutely and uniformly. From (3.1) and (2.8) it follows that the poles of the terms of (2.13) can lie only at the points $p_{k_0, k_1, \ldots, k_l}$ of (3.2) and must be of finite order. Let us consider the vector function

$$H_j(p) = F_j(p)(p - p_{k_0, k_1, \ldots, k_l})^r \tag{3.4}$$

where r is a sufficiently large integer.

When $\varepsilon > 0$ is small enough, the vector $H_j(p)$ is regular in the circle $C_\varepsilon \; |p - p_{k_0, k_1, \ldots, k_l}| \leqslant \varepsilon$, and the sequence $H_j(p)$ converges on the boundary of the circle C_ε uniformly. By Weierstrass' theorem, the sequence $H_j(p)$ converges uniformly inside the circle C_ε, and the coefficients of Taylor's expansion of $H_j(p)$ at the point, $p = p_{k_0, k_1, \ldots, k_l}$ converge to definite finite values. We have thus established the following theorem.

Theorem 3.1. Suppose that in the system (1.1) $\alpha_0 \equiv 0$, Re $\alpha_q > 0$ ($q = 1, \ldots, l$), and $\Phi(t) \equiv 0$. In this case the Laplace transform $F(p)$ of $Y(t)$, given by (2.1), is representable in the form of the series (2.13). The meromorphic vector $F(p)$ can have poles, of finite multiplicity, only at the points $p_{k_0, k_1, \ldots, k_l}$ determined by (3.2). The coefficients of Laurent's expansion of $F(p)$ at the points $p = p_{k_0, k_1, \ldots, k_l}$ converge to the corresponding coefficients of the expansion of the vector $F(p)$.

The last assertion of Theorem 3.1 permits one to obtain the expansion of $Y(t)$ as an asymptotic series for large values of t

$$Y(t) \sim \sum_{k_0, k_1, \ldots, k_l = 0}^{\infty} \operatorname{res}(F(p)e^{pt})|_{p = p_{k_0, k_1, \ldots, k_l}} \tag{3.5}$$

From the properties of the Laplace transform [2], we now obtain the following asymptotic result

$$\Bigl(Y(t) - \sum_{\operatorname{Re} p_{k_0, k_1, \ldots, k_l} > b} \operatorname{res}(F(p)e^{pt})|_{p = p_{k_0, k_1, \ldots, k_l}}\Bigr) e^{-bt} \underset{t \to \infty}{\longrightarrow} 0 \tag{3.6}$$

Theorem (3.1) and property (3.5) imply the next theorem.

Theorem 3.2. Suppose that in the system (1.1)

$$\alpha_0 \equiv 0, \qquad \operatorname{Re} \alpha_q > 0 \qquad (q = 1, \ldots, l)$$

Then we have the following results:

(1) The solutions of the system (1.1) are asymptotically stable if $\operatorname{Re} \rho_k < 0$ $(k = 0, 1, \ldots)$.

(2) The solutions of the system (1.1) are not stable if $\operatorname{Re} \rho_{k_0} > 0$ for at least one ρ_{k_0}

(3) Suppose that $\operatorname{Re} \rho_k \leqslant 0$ $(k = 0, 1, 2, \ldots)$. The solutions of the system (1.1) will be stable if and only if the elements of the matrix $L_0^{-1}(p)$ (2.3) have simple poles for all roots ρ_k lying on the imaginary axis.

The conclusions of Theorem 3.2 can be reformulated in the following way.

Theorem 3.3. Suppose that in the system of Equations (1.1) $\alpha_0 \equiv 0$, $\operatorname{Re} \alpha_q > 0$ $(q = 1, \ldots, l)$, and $\Phi(t) \equiv 0$. In order that all solutions of the system (1.1) may be stable, it is necessary and sufficient that all solutions of the abbreviated system of (1.1), that is, of

$$A_{0n} \frac{d^n Y(t)}{dt^n} + \sum_{k=0}^{n-1} \int_{-h}^{0} dA_{0k}(\vartheta) \frac{d^k Y(t+\vartheta)}{dt^k} = 0 \tag{3.7}$$

be stable.

Note 3.1. If one considers the solutions of the system (1.1) and (3.7) with the initial conditions (1.7), then one can distinguish between their behaviors at infinity, when $t \to +\infty$. For example, the solution of the system (1.1) will tend to zero while the solution of the system of Equations (3.7), with the same initial conditions, will be unbounded.

Note 3.2. In the Theorems 3.2 and 3.3 the main assertions follow already from known results (see, for example, [4]).

If $\Phi(t)$ in (1.1) has the form (1.6), then it follows from (2.13), (2.8), (2.4) and (1.6) that the vector $F(p)$ will have additional poles at the points

$$\omega_{j, k_1, \ldots, k_l} = \omega_j - k_1\alpha_1 - k_2\alpha_2 - \ldots - k_l\alpha_l \qquad (j = 1, \ldots, \lambda,\ k_q = 0, 1, 2, \ldots) \tag{3.8}$$

Example 3.1. Let us consider the differential equation

$$\frac{dy(t)}{dt}+ay(t)+be^{-t}y(t-\tau)=0, \qquad y(0)=1, \qquad y(t)\equiv 0 \qquad (t<0) \tag{3.9}$$

By (2.3) and (2.4) we have

$$L_0(p)=p+a, \qquad L_1(p)=be^{-p\tau}, \qquad R(p)=1 \tag{3.10}$$

For the image $f(p)$ of the solution $y(t)$ of the Equation (3.9), we obtain

$$f(p)=(p+a)^{-1}-be^{(-p+1)\tau}(p+a)^{-1}f(p+1) \tag{3.11}$$

The series (2.13) takes on the form

$$f(p)=\frac{1}{p+a}-\frac{be^{-(p+1)\tau}}{(p+a)(p+a+1)}+\frac{b^2e^{-(2p-3)\tau}}{(p+a)(p+a+1)(p+a+2)}+\dots \tag{3.12}$$

From (3.12) we obtain the asymptotic expansion (3.5)

$$y(t)\sim e^{-at}\left(1-\frac{be^{\tau(a-1)}}{1!}+\frac{b^2e^{\tau(2a+3)}}{2!}-\dots+(-1)^n\frac{b^ne^{\tau(na-n(n+1)/2)}}{n!}+\dots\right)\times$$
$$\times\left(1+\frac{be^{\tau(-a)}}{1!}e^{-t}+\frac{b^2e^{\tau(1-2a)}}{2!}e^{-2t}+\dots+\frac{b^ne^{\tau(n(n-1)/2-na)}}{n!}e^{-nt}+\dots\right) \tag{3.13}$$

Example 3.2. Let us find the asymptotic expansion (3.5) of the solution $y(t)$ of the equation

$$dy/dt+y(t-\pi)e^{-t}=0, \qquad y(t)=\sin t, \qquad t\in[-\pi,0] \tag{3.14}$$

For the image $f(p)$ of the solution $y(t)$ we now have the difference equation

$$f(p)=(e^{-\pi(p+1)}+1)[(p+1)^2+1]p^{-1}-p^{-1}e^{-\pi(p+1)}f(p+1) \tag{3.15}$$

From Formulas (2.13) and (3.5) we obtain the expansion

$$y(t)\sim\left(\frac{1+e^{-\pi}}{(1+1^2)}-\frac{(1+e^{-2\pi})e^{-\pi}}{(1+2^2)1!}+\dots+(-1)^{n+1}\frac{(1+e^{-n\pi})e^{-n(n-1)\pi/2}}{(1+n^2)(n-1)!}+\dots\right)\times$$
$$\times\left(1+e^{-t}+\frac{e^{\pi}}{2!}e^{-2t}+\frac{e^{3\pi}}{3!}e^{-3t}+\dots+\frac{e^{n(n-1)\pi/2}}{n!}e^{-nt}+\dots\right) \tag{3.16}$$

The series (3.16), as well as the series (3.13), diverges for all values of t, but it describes very well the asymptotic behavior of $y(t)$ when $t\to\infty$. For example, if we take the first five terms of the series in the first set of parentheses in (3.16), then we obtain the limiting value $y(+\infty)$ with an accuracy of up to 10^{-22}. By making use of another grouping in finding the original function, one can show that the corresponding series for $y(t)$ converges for all finite values of t. Thus, if one is seeking the original $y(t)$ in Example 3.1 in the form of a power series in b, then one obtains the usual solution, and the expansion will converge for all finite values of t.

4. In order to explain the term "regular case" we note that the problem of the construction of the solution of a system of linear differential equations in the neighborhood of a regular singular point is usually reduced, by means of a simple substitution, to the regular system (1.1) but without a delay in the argument [1]. The methoa which was proposed in Section 3 is one of the most cenvenient ones for the construction of the solution of a system of linear differential equations in the neighborhood of a regular singular point. This is especially so for the various critical cases.

Example 4.1. Let us find the fundamental normalized matrix of the solutions of a system of differential equations (which is frequently met in control problems)

$$x\, dZ/dx = (A - xB)\, Z(x), \qquad Z(1) = E, \qquad x \in [1, 0) \tag{4.1}$$

in the neighborhood of a regular singular point $x = 0$. A change of variables with the aid of the formulas $x = e^{-t}$, $Z(x) \equiv Y(t) = Y(-\ln x)$, leads to the system of equations

$$dY/dt + AY(t) = e^{-t} BY(t), \qquad Y(0) = E, \qquad t \in [0, \infty) \tag{4.2}$$

The system of difference equations (2.2) takes on the form

$$(Ep + A)\, F(p) = E + BF(p+1) \tag{4.3}$$

We obtain its solution in the form

$$F(p) = (Ep + A)^{-1} + (Ep + A)^{-1} B\, (E\,(p+1) + A)^{-1} + \\ + (Ep + A)^{-1} B\, (E\,(p+1) + A)^{-1} B\, (E\,(p+2) + A)^{-1} + \ldots \tag{4.4}$$

Let us denote by $p_1,\ p_2,\ \ldots,\ p_n$ the roots of the equation Det $(Ep + A) = 0$, and let us consider the most simple case when

$$p_j - p_h \neq k \qquad (j, h = 1, 2, \ldots, m;\ \ j \neq h;\ \ k = 0, \pm 1, \pm 2, \ldots) \tag{4.5}$$

Suppose that we have the simplest tyep of partial fractions expansion

$$(Ep + A)^{-1} = C_1\,(p - p_1)^{-1} + C_2\,(p - p_2)^{-1} + \ldots + C_m\,(p - p_m)^{-1} \tag{4.6}$$

We shall denote the noncommutative product of matrices in the usual way

$$\prod_{k=1}^{n} A_k = A_1, A_2, \ldots, A_n \tag{4.7}$$

We can find the original $Y(t)$ with the aid of (3.5), and recalling the change of variables, we obtain

$$Z(x) = \sum_{j=1}^{m} x^{-p_j} \left\{E + \sum_{k=1}^{\infty} x^k \prod_{s=1}^{k} [(E(p_j - k + 1 + s) + A)^{-1}B]\right\} C_j \times$$
$$\times \left\{E + \sum_{k=1}^{\infty} \prod_{s=1}^{k} [B(E(p_j + s) + A)^{-1}]\right\} \tag{4.8}$$

Example 4.2. Let us find the solution, when $x \in [1, 0)$, of a differential equation which has a regular singular point at $x = 0$

$$x\frac{d^2z(x)}{dx^2} + \frac{dz(x)}{dx} - z(x) = 0, \qquad \frac{dz}{dx}(1) = -1, \qquad z(1) = 0 \tag{4.9}$$

By means of the change of variables $x = e^{-t}$, $t \in [0, \infty)$, $z(x) \equiv y(t)$, we obtain

$$\frac{d^2y(t)}{dt^2} - e^{-t}y(t) = 0, \qquad y(0) = 0, \qquad \frac{dy}{dt}(0) = 1 \tag{4.10}$$

The difference equation (2.2) and its solution (2.13) have the forms

$$p^2 f(p) = 1 + f(p+1), \quad f(p) = \frac{1}{p^2} + \frac{1}{p^2(p+1)^2} + \frac{1}{p^2(p+1)^2(p+2)^2} + \ldots \tag{4.12}$$

Here we have the critical case for the differential equation (4.10), when the image has a pole of order higher than the first. We are looking for the original by the usual rules, determining the principal part of the expansion of $f(p)$ in terms of the poles $p = 0, -1, -2, \ldots$ Introducing two constants a and b, we obtain the solution in the form of the series

$$y(t) = (at + 2b)\sum_{k=0}^{\infty} \frac{e^{-kt}}{(k!)^2} + 2a\sum_{k=1}^{\infty} \frac{e^{-kt}}{(k!)^2}\sum_{s=1}^{k}\frac{1}{s} \qquad \left(a = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\right) \tag{4.13}$$

$$z(x) = (2b - a\ln x)\sum_{k=0}^{\infty}\frac{x^k}{(k!)^2} + 2a\sum_{k=1}^{\infty}\frac{x^k}{(k!)^2}\sum_{s=1}^{k}\frac{1}{s} \qquad \left(b = \sum_{k=1}^{\infty}\frac{1}{(k!)^2}\left(\sum_{s=1}^{k}\frac{1}{s}\right)\right) \tag{4.14}$$

BIBLIOGRAPHY

1. Valeev, K.G., O reshenii i kharakteristicheskikh pokazateliakh reshenii nekotorykh sistem lineinykh differentsial'nykh uravnenii s periodicheskimi koeffitsientami (On the solution and characteristic exponents of solutions of some systems of linear differential equations with periodic coefficients). *PMM* Vol. 24, No. 4, 1960.

2. Levrent'ev, M.A. and Shabat, B.V., *Metody teorii funktsii kompleksnogo peremennogo* (*Methods of the Theory of Functions of a Complex Variable*). Fizmatgiz, 1958.

3. Shilov, G.E., *Matematicheskii analiz. Spetsial'nyi kurs* (*Mathematical Analysis. Special Course*). Fizmatgiz, 1960.

4. Krasovskii, N.N., *Nekotorye zadachi teorii ustoichivosti dvizheniia* (*Some Problems of the Theory of Stability of Motion*). GIFML, 1959.

Translated by H.P.T.